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Unique Continuous Selections for Metric Projections of $C(X)$ onto Finite-Dimensional Vector Subspaces, II

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Best approximation in $C(X)$ by elements of a Chebyshev subspace is governed by Haar's theorem, the de la Vallée Poussin estimates, the alternation theorem, the Remez algorithm, and Mairhuber's theorem. J. Blatter (1990, *J. Approx. Theory* **61**, 194–221) considered best approximation in $C(X)$ by elements of a subspace whose metric projection has a unique continuous selection and extended Haar's theorem and Mairhuber's theorem to this situation. In the present paper we so extend the de la Vallée Poussin estimates, the alternation theorem, and the Remez algorithm.

INTRODUCTION

Throughout this paper we deal with best approximation of elements of the space $C(X)$ of all continuous real-valued functions on a compact Hausdorff topological space X in the *uniform norm*

$$\|f\| = \sup\{|f(x)| : x \in X\}, \qquad f \in C(X),$$

by elements of a vector subspace G of finite dimension $n \geq 1$. For $f \in C(X)$, the *distance of f to G* is the non-negative real number

$$d(f) = \inf\{\|f-g\| : g \in G\},$$

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and the *set of best approximations of f in G* is the non-empty compact convex subset

$$P(f) = \{g \in G : \|f - g\| = d(f)\}$$

of G. The (set-valued) *metric projection of* ($C(X)$ onto) G is the mapping P of $C(X)$ into the power set of G which maps $f \in C(X)$ onto $P(f)$, and a *continuous selection of the metric projection of* G is a continuous mapping S of $C(X)$ into G with the property that $Sf \in P(f)$ for every $f \in C(X)$.

G is called a *Chebyshev subspace of* $C(X)$ if every $f \in C(X)$ has a unique best approximation in G; it is part of the folklore of the subject that in this case the metric projection of G, considered as a mapping of $C(X)$ into G, is continuous. A. Haar [6] gave the following intrinsic description of such G.

HAAR'S THEOREM. *G is a Chebyshev subspace of $C(X)$ iff any non-zero function in G has at most $n-1$ distinct zeros.*

Best approximation in $C(X)$ by elements of a Chebyshev subspace G is governed by Haar's theorem, the de la Vallée Poussin estimates, the alternation theorem, the Remez algorithm, and Mairhuber's theorem. J. Blatter [1] considered best approximation in $C(X)$ by elements of a subspace G whose metric projection has a unique continuous selection and showed that Haar's theorem has the following extension to this situation.

BLATTER'S THEOREM. *The metric projection of G has a unique continuous selection iff*

(1) *any non-zero function in G has at most n distinct zeros;*

(2) *for any $1 \leqslant m \leqslant n$ distinct isolated points $x_1, ..., x_m$ of X,*

$$\dim\{g \in G : g(x_1) = \cdots = g(x_m) = 0\} \leqslant n - m; \text{ and}$$

(3) *for any n distinct points $x_1, ..., x_n$ of X and any n signs $s_1, ..., s_n$ in $\{-1, 1\}$, there exists a non-zero function g in G with the property that for each $i = 1, ..., n$ the function $s_i g$ is non-negative in a neighborhood of x_i.*

In the same paper Blatter also extended Mairhuber's theorem to the new situation. In the present paper we so extend the de la Vallée Poussin estimates, the alternation theorem, and the Remez algorithm. This is done in Sections 2 and 3.

G is called an *almost Chebyshev subspace of* $C(X)$ (A. L. Garkavi [4]) if the set of functions in $C(X)$ which do not have a unique best approximation in G is of the first category in $C(X)$. A. L. Garkavi [4; Theorem I, and last paragraph on p. 186 of the English translation] gave the following intrinsic description of such G.

Garkavi's Theorem. *G is an almost Chebyshev subspace of $C(X)$ iff for any non-zero function $g \in G$,* $\operatorname{card} \operatorname{int} Z(g) \leqslant n-1$ (card = cardinal number of, int = interior of, $Z(g)$ = the zero set of g) *and for any $1 \leqslant m \leqslant n-1$ distinct isolated points $x_1, ..., x_m$ of X,*

$$\dim\{g \in G : g(x_1) = \cdots = g(x_m) = 0\} \leqslant n-m.$$

Garkavi's theorem shows that in the presence of condition (1) in Blatter's theorem, condition (2) is equivalent to the condition that G be an almost Chebyshev subspace of $C(X)$. Thus, if we agree to call G a *weakly interpolating subspace of* $C(X)$ (F. Deutsch and G. Nürnberger [3]) if G satisfies condition (3), we may restate Blatter's theorem in the following slightly redundant form.

The metric projection of G has a unique continuous selection iff G is a weakly interpolating almost Chebyshev subspace of $C(X)$ with the property that $\operatorname{card} Z(g) \leqslant n$ for every $g \in G \sim \{0\}$.

In Section 1 of the present paper we show that a weakly interpolating almost Chebyshev subspace G of $C(X)$ is a natural habitat for "σ-alternators." These "σ-alternators" are the key to the results in Sections 2 and 3.

In order to render this paper reasonably self-contained without cluttering it up, we found it convenient to gather some simple but not quite obvious results of a rather general nature which we use frequently, in an appendix.

1. σ-Alternators Defined

In the sequel we assume that $g_1, ..., g_n$ is a fixed basis for G. We define a function $v: X \to \mathbf{R}^n$ by

$$v(x) = (g_1(x), ..., g_n(x)), \qquad x \in X,$$

set

$$\Delta_n = \{(x_1, ..., x_n) \in X^n : \text{two of the } x_i \text{ coincide}\},$$

define a function $D: X^n \sim \Delta_n \to \mathbf{R}$ by

$$D(p) = \det(v(x_1), ..., v(x_n)), \qquad p = (x_1, ..., x_n) \in X^n \sim \Delta_n,$$

and note that a change of the basis $g_1, ..., g_n$ for G amounts to multiplication of D by a non-zero constant.

We adopt the following notation: For a non-empty closed subset Y of X and for $f \in C(X)$, the *norm on Y of f* is

$$\|f\|_Y = \sup\{|f(x)| : x \in Y\},$$

the *distance on Y of f to G* is

$$d_Y(f) = \inf\{\|f-g\|_Y : g \in G\},$$

and the *set of best approximations on Y of f in G* is

$$P_Y(f) = \{g \in G : \|f-g\|_Y = d_Y(f)\}.$$

LEMMA 1. *These conditions on G are equivalent.*

(a) *G is an almost Chebyshev subspace of $C(X)$.*

(b) *For any* $g \in G \sim \{0\}$, $\operatorname{card} \operatorname{int} Z(g) \leqslant n-1$, *and for any* $1 \leqslant m \leqslant n-1$ *distinct isolated points* $x_1, ..., x_m$ *of* X,

$$\dim\{g \in G : g(x_1) = \cdots = g(x_m) = 0\} \leqslant n-m.$$

(b′) *For any* $g \in G \sim \{0\}$,

$$\operatorname{card} \operatorname{int} Z(g) \leqslant n - \dim\{h \in G : h = 0 \text{ on } \operatorname{int} Z(g)\}.$$

(c) *The set* $\{p \in X^n \sim \Delta_n : D(p) \neq 0\}$ *is dense in* $X^n \sim \Delta_n$.

(c′) *For any* $N \geqslant n$ *distinct points* $x_1, ..., x_N$ *of* X *and any disjoint neighbourhoods* U_i *of the* x_i *there exist points* $y_i \in U_i$, $i = 1, ..., N$, *such that* $D(y_{i_1}, ..., y_{i_n}) \neq 0$ *for any n distinct indices* $1 \leqslant i_1, ..., i_n \leqslant N$; *in other words,* $G \mid \{y_1, ..., y_N\}$ ($\mid$ = restricted to) *is n-dimensional and satisfies the Haar condition.*

(c″) *For any member U of the uniformity $\mathscr{U}$ of X* (for all uniform notions employed, refer to the uniformity of X in the Appendix) *there exists a finite U-net Y in X with the property that $G \mid Y$ is n-dimensional and satisfies the Haar condition.*

Proof. The equivalence of (a) and (b) is, of course, Garkavi's theorem, the equivalence of (b) and (b′) was observed in J. Blatter [1], and that (b) implies (c) was stated without proof in *A*. L. Garkavi [4]; for a proof see J. Blatter [1].

(c) $\Rightarrow$ (c′). Suppose (c) holds and suppose we are given $N \geqslant n$ distinct points $x_1, ..., x_N$ of X and disjoint *open* neighbourhoods U_i of the x_i. We may and shall suppose that $N \geqslant n+1$.

Let

$$\left\{1, ..., n!\binom{N}{n}\right\} \xrightarrow{\varphi} \{(i_1, ..., i_n) : 1 \leqslant i_1, ..., i_n \leqslant N \text{ distinct}\}$$

be any bijection, and suppose for a moment that for each $k=1, ..., n!\binom{N}{n}$ we have constructed non-empty open subsets $V_{1,k}, ..., V_{N,k}$ of $U_1, ..., U_N$, respectively, with the property that

$$\text{if } 1 \leqslant l \leqslant k \leqslant n!\binom{N}{n} \text{ with, say, } \varphi(l)=(i_1, ..., i_n), \text{ then } D(y_{i_1}, ..., y_{i_n}) \neq 0$$

$$\text{for any } (y_{i_1}, ..., y_{i_n}) \in V_{i_1,k} \times \cdots \times V_{i_n,k}. \tag{$*$}$$

It is clear then that any points

$$y_i \in V_{i,n!\binom{N}{n}}, \qquad i=1, ..., N,$$

have the required property. We now construct the $V_{1,k}, ..., V_{N,k}$ by induction over k.

Let $\varphi(1)=(j_1, ..., j_n)$. By (c) and by the continuity of D, there exist non-empty open subsets $V_{j_1,1}, ..., V_{j_n,1}$ of $U_{j_1}, ..., U_{j_n}$, respectively, such that

$$D(y_{j_1}, ..., y_{j_n}) \neq 0 \qquad \text{for any} \quad (y_{j_1}, ..., y_{j_n}) \in V_{j_1,1} \times, ..., \times V_{j_n,1}.$$

Set $V_{j,1}=U_j$ for all $j \in \{1, ..., N\} \sim \{j_1, ..., j_n\}$.

Now suppose we have constructed $V_{1,k}, ..., V_{N,k}$ with the property $(*)$ for some $1 \leqslant k < n!\binom{N}{n}$. Let $\varphi(k+1)=(j_1, ..., j_n)$. Again by (c) and by the continuity of D there exist non-empty open subsets $V_{j_1,k+1}, ..., V_{j_n,k+1}$ of $V_{j_1,k}, ..., V_{j_n,k}$, respectively, such that

$$D(y_{j_1}, ..., y_{j_n}) \neq 0 \qquad \text{for any} \quad (y_{j_1}, ..., y_{j_n}) \in V_{j_1,k+1} \times \cdots \times V_{j_n,k+1}.$$

Set $V_{j,k+1}=V_{j,k}$ for all $j \in \{1, ..., N\} \sim \{j_1, ..., j_n\}$.

(c′) $\Rightarrow$ (c″). Suppose (c′) holds and suppose that $U \in \mathscr{U}$. There exists a symmetric $V \in \mathscr{U}$ such that $V \circ V = \{(x, y) : (x, z), (z, y) \in V$ for some $z\} \subset U$ and there exists a finite V-net $\{x_1, ..., x_N\}$ in X. We may and shall suppose that $N \geqslant n$. By (c′) there exist distinct points $y_i \in V[x_i]$, $i=1, ..., N$, such that $G \mid \{y_1, ..., y_N\}$ is n-dimensional and satisfies the Haar condition. Now let $x \in X$. Since $\{x_1, ..., x_N\}$ is a V-net, $x \in V[x_i]$ for some i. Since V is symmetric and $V \circ V \subset U$, $x \in U[y_i]$. Thus $\{y_1, ..., y_N\}$ is a U-net. Set $Y=\{y_1, ..., y_N\}$.

(c″) $\Rightarrow$ (a). Suppose (c″) holds and suppose that $f \in C(X) \sim G$. There exists a sequence $\{U_k\}_{k \in \mathbf{N}}$ in $\mathscr{U}$ such that

$$\lim_{k \in \mathbf{N}} \Omega(f, g_1, ..., g_n; U_k)=0.$$

By (c″), for every $k \in \mathbf{N}$ there exists a finite U_k-net Y_k in X such that $G \mid Y_k$ is n-dimensional and satisfies the Haar condition. Set $P_{Y_k}(f)=\{h_k\}$ for every $k \in \mathbf{N}$. By the first discretization lemma in the Appendix, the sequence $\{d_{Y_k}(f)\}_{k \in \mathbf{N}}$ converges to $d(f)$ and the sequence $\{h_k\}_{k \in \mathbf{N}}$ has a subsequence $\{h_{k_l}\}_{l \in \mathbf{N}}$ which converges to some $h \in P(f)$. For every $l \in \mathbf{N}$, set

$$f_l = (f \vee (h_{k_l} - d_{Y_{k_l}}(f))) \wedge (h_{k_l} + d_{Y_{k_l}}(f)) \qquad (\vee, \wedge = \sup, \inf).$$

The sequence $\{f_l\}_{l \in \mathbf{N}}$ converges to f and, since $\|f_l - h_{k_l}\| = d_{Y_{k_l}}(f)$, $P(f_l) = \{h_{k_l}\}$ for every $l \in \mathbf{N}$.

We have shown that the set of functions in $C(X)$ which have a unique best approximation in G is dense in $C(X)$, and this (see J. Blatter [1]) is enough for G to be an almost Chebyshev subspace of $C(X)$.

LEMMA 2. *If G is an almost Chebyshev subspace of $C(X)$ then for any n disjoint non-empty open subsets $U_1, ..., U_n$ of X the following two conditions are equivalent.*

(a) *There exists a sign $s \in \{-1, 1\}$ such that*

$$sD(p) \geqslant 0 \qquad \text{for all} \quad p \in \prod_{i=1}^{n} U_i.$$

(b) *Given $N \geqslant n+1$ distinct points $x_1, ..., x_N \in \bigcup_{i=1}^n U_i$ with the property that $D(x_{i_1}, ..., x_{i_n}) \neq 0$ for some $1 \leqslant i_1, ..., i_n \leqslant N$, and given non-zero real numbers $\alpha_1, ..., \alpha_N$ with the property that* $\operatorname{sgn} \alpha_i = \operatorname{sgn} \alpha_j$ (sgn = sign of) *whenever x_i and x_j belong to the same U_k, there exists a $g \in G$ such that $\sum_{i=1}^N \alpha_i g(x_i) \neq 0$.*

Proof. (a) $\Rightarrow$ (b). Suppose (a) holds and (b) does not. Then there exist $N \geqslant n+1$ distinct points $x_1, ..., x_N \in \bigcup_{i=1}^n U_i$ with the property that $D(x_{i_1}, ..., x_{i_n}) \neq 0$ for some $1 \leqslant i_1, ..., i_n \leqslant N$, and there exist non-zero real numbers $\alpha_1, ..., \alpha_N$ with the property that $\operatorname{sgn} \alpha_i = \operatorname{sgn} \alpha_j$ whenever x_i and x_j belong to the same U_k, such that $\sum_{i=1}^N \alpha_i g(x_i) = 0$ for all $g \in G$.

Set $I = \{i \in \{1, ..., n\} : x_j \in U_i \text{ for some } j \in \{1, ..., N\}\}$, set $m = \operatorname{card} I$, and if $m < n$ choose for each $i \in \{1, ..., n\} \sim I$ an arbitrary point $y_i \in U_i$.

Set $J_i = \{j \in \{1, ..., N\} : x_j \in U_i\}$ for every $i \in I$, and use the second fact about $\mathbf{R}^n$ in the Appendix ($\sum_{i=1}^N |\alpha_i| (\operatorname{sgn} \alpha_i v(x_i)) = 0!$) and the implication "(a) $\Rightarrow$ (c′)" in Lemma 1 to obtain distinct points x_i^*, $i \in \{1, ..., N\}$, and y_i^*, $i \in \{1, ..., n\} \sim I$, of X and non-zero real numbers $\alpha_1^*, ..., \alpha_N^*$ such that

- if $i \in I$ and $j \in J_i$ then $x_j^* \in U_i$ and $\operatorname{sign} \alpha_j^* = \operatorname{sign} \alpha_j$;
- $\sum_{i=1}^N \alpha_i^* v(x_i^*) = 0$;
- if $i \in \{1, ..., n\} \sim I$ then $y_i^* \in U_i$; and
- $D(p) \neq 0$ for any point $p \in X^n \sim \Delta_n$ with coordinates only among the x_i^* and y_i^*.

For every $(j_1, ..., j_m) \in \prod_{i \in I} J_i$ let $p_{(j_1, ..., j_m)}$ be the point of $X^n \sim \Delta_n$ whose ith coordinate is

$$\begin{cases} x_{j_k}^* & \text{if} \quad i \in I \quad \text{and} \quad j_k \in J_i \\ y_i^* & \text{if} \quad i \in \{1, ..., n\} \sim I. \end{cases}$$

For every $i \in \{1, ..., n\}$ set

$$v_i = \begin{cases} \sum_{j \in J_i} \alpha_j^* v(x_j^*) & \text{if} \quad i \in I \\ v(y_i^*) & \text{if} \quad i \in \{1, ..., n\} \sim I \end{cases}$$

and for every $i \in I$ let s_i be the common sign of the α_j^*, $j \in J_i$.

The identity

$$\sum_{(j_1, ..., j_m) \in \prod_{i \in I} J_i} |\alpha_{j_1}^* \cdots a_{j_m}^*| \, sD(p_{(j_1, ..., j_m)}) = s_1 \cdots s_m s \det(v_1, ..., v_n)$$

is now obvious. By (a) and by the construction of the x_i^*, y_i^*, and α_i^*, all the terms of the sum on the left are positive. Since $\sum_{i \in I} v_i = 0$, the determinant on the right is zero. We have reached a contradiction.

(b) $\Rightarrow$ (a). Suppose (b) holds. By the implication "(a) $\Rightarrow$ (c)" in Lemma 1 there exists a point $p = (x_1, ..., x_n) \in \prod_{i=1}^n U_i$ such that $D(p) \neq 0$. Set $s = \operatorname{sgn} D(p)$. Since D is continuous there exist open neighborhoods $V_1, ..., V_n$ of $x_1, ..., x_n$, respectively, such that $V_i \subset U_i$ for all i and $sD(q) > 0$ for all $q \in \prod_{i=1}^n V_i$. We show by induction over $k = 0, ..., n$ that

$$sD(q) > 0 \text{ whenever } q = (y_1, ..., y_n) \in X^n \sim \Delta_n \text{ is such that}$$
$$y_i \in U_i \text{ if } 1 \leqslant i \leqslant k \text{ and } y_i \in V_i \text{ if } k+1 \leqslant i \leqslant n. \tag{$*$}$$

By our choice of the V_i, $(*)$ holds for $k = 0$. Suppose then that $(*)$ holds for some $0 \leqslant k < n$ and suppose that $sD(q) < 0$ for some $q = (y_1, ..., y_n) \in X^n \sim \Delta_n$ such that $y_i \in U_i$ if $1 \leqslant i \leqslant k+1$ and $y_i \in V_i$ if $k+2 \leqslant i \leqslant n$. By our hypotheses, $y_{k+1} \in U_{k+1} \sim \operatorname{cl} V_{k+1}$ (cl = closure of). Choose $y_{n+1} \in V_{k+1}$ and use the continuity of D and the implication "(a) $\Rightarrow$ (c′)" in Lemma 1 to obtain $z_1, ..., z_{n+1} \in X$ such that

- $z_i \in U_i$ if $1 \leqslant i \leqslant k$, $z_{k+1} \in U_{k+1} \sim \operatorname{cl} V_{k+1}$, $z_i \in V_i$ if $k+2 \leqslant i \leqslant n$ and $z_{n+1} \in V_{k+1}$;
- $D(z_1, ..., \widehat{z_i}, ..., z_{n+1}) \neq 0$ for $i = 1, ..., n$ ($\widehat{}$ = omit what is under it); and
- $sD(z_1, ..., z_n) < 0$.

Obviously, there exist $\alpha_1, ..., \alpha_{n+1} \in \mathbf{R}$ not all zero such that

$\sum_{i=1}^{n+1} \alpha_i v(z_i) = 0$. By the third fact about $\mathbf{R}^n$ in the Appendix, there exists a $\gamma \in \mathbf{R} \sim \{0\}$ such that

$$\alpha_i = \gamma(-1)^i D(z_1, ..., \widehat{z_i}, ..., z_{n+1}) \qquad \text{for} \quad i = 1, ..., n+1.$$

Thus, all of the α_i are different from zero. Since $sD(z_1, ..., z_n) < 0$,

$$\operatorname{sgn} \alpha_{n+1} = \operatorname{sgn} \gamma(-1)^{n+1} \operatorname{sgn} D(z_1, ..., z_n) = \operatorname{sgn} \gamma(-1)^n s,$$

and since, by the induction hypothesis, $sD(z_1, ..., z_k, z_{n+1}, z_{k+2}, ..., z_n) > 0$,

$$\begin{aligned} \operatorname{sgn} \alpha_{k+1} &= \operatorname{sgn} \gamma(-1)^{k+1} \operatorname{sgn} D(z_1, ..., \widehat{z_{k+1}}, ..., z_{n+1}) \\ &= \operatorname{sgn} \gamma(-1)^{k+1} (-1)^{n+k+1} D(z_1, ..., z_k, z_{n+1}, z_{k+2}, ..., z_n) \\ &= \operatorname{sgn} \gamma(-1)^n s. \end{aligned}$$

Thus, $\operatorname{sgn} \alpha_{n+1} = \operatorname{sgn} \alpha_{k+1}$. This contradicts (b), whence (*) holds for $k = n$, and this is just (a).

COROLLARY. *G is a weakly interpolating almost Chebyshev subspace of $C(X)$ iff $X^n \sim \Delta_n$ is the disjoint union of the closures (in $X^n \sim \Delta_n$!) of the sets*

$$\operatorname{pos}(D) = \{p \in X^n \sim \Delta_n : D(p) > 0\} \qquad \textit{and}$$
$$\operatorname{neg}(D) = \{p \in X^n \sim \Delta_n : D(p) < 0\},$$

in symbols,

$$X^n \sim \Delta_n = \operatorname{cl} \operatorname{pos}(D) \mathbin{\dot\cup} \operatorname{cl} \operatorname{neg}(D).$$

Proof. Fix any n disjoint non-empty open subsets $U_1, ..., U_n$ of X. By the first fact about $\mathbf{R}^n$ in the Appendix, condition (b) in Lemma 2 is equivalent to the condition

$$0 \notin \operatorname{int} \operatorname{conv} \left(\bigcup_{i=1}^n s_i v[U_i] \right) \qquad \text{for any } n \text{ signs } s_1, ..., s_n \in \{-1, 1\}.$$

Now fix $s_1, ..., s_n \in \{-1, 1\}$. By your favourite separation theorem,

$$0 \notin \operatorname{int} \operatorname{conv} \left(\bigcup_{i=1}^n s_i v[U_i] \right) \qquad \text{iff there exists a } c \in \mathbf{R}^n \sim \{0\}$$

$$\text{such that } \langle c, a \rangle \geqslant 0 \quad \text{for all } a \in \bigcup_{i=1}^n s_i v[U_i] \quad (\langle \cdot, \cdot \rangle = \text{scalar product}).$$

Finally, fix $c=(c_1, ..., c_n) \in \mathbf{R}^n \sim \{0\}$ and set $g=\sum_{i=1}^n c_i g_i$. Obviously

$$\langle c, a\rangle \geqslant 0 \qquad \text{for all} \quad a \in \bigcup_{i=1}^{n} s_i v[U_i]$$

$$\text{iff } s_i g(x) \geqslant 0 \qquad \text{for all} \quad i=1, ..., n \quad \text{and all} \quad x \in U_i.$$

The Corollary now follows from Lemmas 1 and 2.

DEFINITION. Suppose that G is a weakly interpolating almost Chebyshev subspace of $C(X)$.

Appealing to the Corollary, we define a *sign function*

$$\sigma : X^n \sim \Delta_n \rightarrow \{-1, 1\}$$

for the function D by

$$\sigma(p)=\begin{cases} 1 & \text{if} \quad p \in \operatorname{cl} \operatorname{pos}(D), \\ -1 & \text{if} \quad p \in \operatorname{cl} \operatorname{neg}(D), \end{cases} \qquad p \in X^n \sim \Delta_n.$$

We set

$$\Delta_{n+1}=\{(x_1, ..., x_{n+1}) \in X^{n+1} : \text{two of the } x_i \text{ coincide}\},$$

define a *reference in* X to be any point of $X^{n+1} \sim \Delta_{n+1}$, and set, for any reference $R=(x_1, ..., x_{n+1})$ in X,

$$\left.\begin{aligned} D_{R,i}&=D(x_1, ..., \widehat{x_i}, ..., x_{n+1}) \\ \sigma_{R,i}&=\sigma(x_1, ..., \widehat{x_i}, ..., x_{n+1}) \end{aligned}\right\} \qquad \text{for} \quad i=1, ..., n+1.$$

For a function $f \in C(X) \sim G$, a *σ-alternator of f in G* is a function $g \in G$ with the property that for some reference $R=(x_1, ..., x_{n+1})$ in X and for some sign $s \in \{-1, 1\}$,

$$(f-g)(x_i)=s(-1)^i \sigma_{R,i} \|f-g\| \qquad \text{for} \quad i=1, ..., n+1$$

(see M. Sommer [10, 11]).

We note that the concept of a σ-alternator is independent of the particular basis for G used in its definition (see the note on D at the beginning of this section) and also that it is permutation invariant as it should be: If $R=(x_1, ..., x_{n+1})$ is a reference in X, if π is an element of the permutation group of order $n+1$, and if R_π is the permuted reference $(x_{\pi(1)}, ..., x_{\pi(n+1)})$, then, representing π as a product of transpositions and using induction over the number of transpositions, one easily sees that

$$(-1)^i \sigma_{R_\pi, i}=\operatorname{sgn} \pi(-1)^{\pi(i)} \sigma_{R, \pi(i)} \qquad \text{for} \quad i=1, ..., n+1.$$

Remarks. 1. We note that our proof of "(b) $\Rightarrow$ (c) $\Rightarrow$ (c$'$) $\Rightarrow$ (c'') $\Rightarrow$ (a)" in Lemma 1 is a new proof for the difficult half of Garkavi's theorem.

2. The condition (c$''$) in Lemma 1 should be contrasted with the following

EXAMPLE. Let α be an ordinal such that

$$\aleph_\alpha \geqslant 2^{2^{2^{\aleph_0}}},$$

set $X = [0, 1]^{\aleph_\alpha}$, define $g_1, g_2 \in C(X)$ by

$$g_1 = 1, \qquad g_2 = \pi_1 \qquad (=\text{projection onto the first factor}),$$

and set $G = \operatorname{span}\{g_1, g_2\}$. Then G is a 2-dimensional almost Chebyshev subspace of $C(X)$ which on no dense subset of X is 2-dimensional and satisfies the Haar condition.

Proof. By Garkavi's theorem, G is a 2-dimensional almost Chebyshev subspace of $C(X)$. Now suppose that G is 2-dimensional and satisfies the Haar condition on some subset Y of X. It is clear from the definition of G that card $Y \leqslant 2^{\aleph_0}$ and this implies that Y is not dense in X: Were Y dense in X, then (see, e.g. L. Gillman and M. Jerison [5; 9A])

$$\operatorname{card} X \leqslant 2^{2^{\operatorname{card} Y}},$$

and therefore

$$\aleph_\alpha < 2^{\aleph_\alpha} = 2^{\aleph_0 \cdot \aleph_\alpha} = (2^{\aleph_0})^{\aleph_\alpha} = \operatorname{card} X \leqslant 2^{2^{\operatorname{card} Y}} \leqslant 2^{2^{2^{\aleph_0}}},$$

contrary to our choice of α.

3. In the light of the equivalence "(a) $\Leftrightarrow$ (c)" in Lemma 1, the Corollary suggests the question if the condition that G be a weakly interpolating subspace of $C(X)$ is equivalent to the condition that $\operatorname{cl} \operatorname{pos}(D) \cap \operatorname{cl} \operatorname{neg}(D) = \phi$. The answer to this question is "no," as the following example of F. Deutsch and G. Nürnberger [3] shows: Set $X = [-2, 2]$, define $g_1, g_2 \in C(X)$ by

$$g_1(x) = \begin{cases} 0 & \text{for } -2 \leqslant x \leqslant 0, \\ x & \text{for } 0 \leqslant x \leqslant 2, \end{cases} \qquad g_2(x) = 1 - |x| \text{ for } |x| \leqslant 2,$$

and set $G = \operatorname{span}\{g_1, g_2\}$. Then G is a 2-dimensional weakly interpolating subspace of $C(X)$, but $(-1, 1) \in \operatorname{cl} \operatorname{pos}(D) \cap \operatorname{cl} \operatorname{neg}(D)$.

W. Li [9] showed that both the condition that G be a weakly interpolating subspace of $C(X)$ and the condition that $\operatorname{cl} \operatorname{pos}(D) \cap \operatorname{cl} \operatorname{neg}(D) = \phi$ are satisfied whenever the metric projection of G has a continuous selection.

2. Existence and Unicity of σ-Alternators; Unique σ-Alternators = Unique Continuous Selections

For this section we assume that G is a weakly interpolating almost Chebyshev subspace of $C(X)$.

An *admissible reference in* X is a reference $R=(x_1, ..., x_{n+1})$ in X with the property that $\dim(G \mid \{x_1, ..., x_{n+1}\})=n$; another way of saying this is that the vectors $v(x_1), ..., v(x_{n+1})$ span $\mathbf{R}^n$, or then that at least one of the determinants $D_{R,1}, ..., D_{R,n+1}$ is different from zero. By the implication "(a) $\Rightarrow$ (c′)" in Lemma 1, the set of admissible references in X is dense in the set $X^{n+1} \sim \Delta_{n+1}$ of all references in X, and by the equivalence "(a) $\Leftrightarrow$ (b)" in the fact from linear algebra in the Appendix, every reference in X is admissible iff card $Z(g) \leqslant n$ for all $g \in G \sim \{0\}$.

For an admissible reference $R=(x_1, ..., x_{n+1})$ in X, we set

$$\mu_{R,i}=\left(\sum_{j=1}^{n+1} |D_{R,j}|\right)^{-1} (-1)^i D_{R,i} \qquad \text{for} \quad i=1, ..., n+1;$$

by the third fact about $\mathbf{R}^n$ in the Appendix, the numbers $\mu_{R,i}$ are characterized by the equations

$$\sum_{i=1}^{n+1} \mu_{R,i} v(x_i)=0 \qquad \text{and} \qquad \sum_{i=1}^{n+1} (-1)^i \sigma_{R,i} \mu_{R,i}=1.$$

We adopt the following notation: For a non-empty closed subset Y of X, a *reference in* Y is a reference in X whose points all belong to Y, and for $f \in C(X)$ and a non-empty closed subset Y of X such that $d_Y(f)>0$, a *σ-alternator on* Y *of* f *in* G is a function $g \in G$ with the property that for some reference $R=(y_1, ..., y_{n+1})$ in Y and for some sign $s \in \{-1, 1\}$,

$$(f-g)(y_i)=s(-1)^i \sigma_{R,i} \|f-g\|_Y \qquad \text{for} \quad i=1, ..., n+1.$$

The de la Vallée Poussin Estimates. 1. *For any* $g \in G$, *any reference* $R=(x_1, ..., x_{n+1})$ *in* X, *and any sign* $s \in \{-1, 1\}$,

$$\inf\{s(-1)^i \sigma_{R,i} g(x_i) : i=1, ..., n+1\} \leqslant 0.$$

2. *If* $f \in C(X)$ *and* $g \in G$ *are such that*

$$s(-1)^i \sigma_{R,i} (f-g)(x_i)>0, \qquad i=1, ..., n+1,$$

for some reference $R=(x_1, ..., x_{n+1})$ *in* X *and for some sign* $s \in \{-1, 1\}$, *then*

$$d_R(f) \geqslant \inf\{|(f-g)(x_i)| : i=1, ..., n+1\}$$

(*the notation "$d_R(f)$" is slightly abusive*!), *and*

$$\sup\{s(-1)^i \sigma_{R,i}(f-h)(x_i) : i=1, ..., n+1\} > 0 \qquad \textit{for all} \quad h \in G.$$

3. *If $f \in C(X)$, if Y is a non-empty closed subset of X with the property that $d_Y(f) > 0$, and if g is a σ-alternator on Y of f in G, then g is a best approximation on Y of f in G.*

Proof. 1. Suppose that $g \in G$ is such that

$$s(-1)^i \sigma_{R,i}\, g(x_i) > 0, \qquad i=1, ..., n+1,$$

for some reference $R=(x_1, ..., x_{n+1})$ in X and some sign $s \in \{-1, 1\}$. Then, by the continuity of g, by the implication "(a) $\Rightarrow$ (c′)" in Lemma 1, and by the Corollary, there exists an admissible reference $R^* = (x_1^*, ..., x_{n+1}^*)$ in X such that

$$s(-1)^i \sigma_{R^*,i}\, g(x_i^*) > 0, \qquad i=1, ..., n+1,$$

and it follows that

$$0 = \sum_{i=1}^{n+1} \mu_{R^*,i}\, g(x_i^*) = s \sum_{i=1}^{n+1} |\mu_{R^*,i}|\, |g(x_i^*)|,$$

a contradiction.

2. Let f, g, R, and s be as specified, and suppose first that

$$d_R(f) < d = \inf\{|(f-g)(x_i)| : i=1, ..., n+1\}.$$

Then there is an $h \in G$ such that $\|f-h\|_R < d$, and it follows that

$$\begin{aligned} s(-1)^i \sigma_{R,i}(h-g)(x_i) &= s(-1)^i \sigma_{R,i}(f-g)(x_i) \\ &\quad - s(-1)^i \sigma_{R,i}(f-h)(x_i) > 0 \\ &\quad \text{for} \quad i=1, ..., n+1, \end{aligned} \tag{$*$}$$

a contradiction to 1. Now suppose that for some $h \in G$

$$s(-1)^i \sigma_{R,i}(f-h)(x_i) \leqslant 0 \qquad \text{for} \quad i=1, ..., n+1.$$

Then we again have ($*$)—although for different reasons— and ($*$) still contradicts 1.

3. Let f, Y, and g be as specified, say,

$$(f-g)(y_i) = s(-1)^i \sigma_{R,i}\, \|f-g\|_Y, \qquad i=1, ..., n+1,$$

for some reference $R=(y_1, ..., y_{n+1})$ in Y and for some sign $s \in \{-1, 1\}$.

Then, by 2,

$$\begin{aligned}\|f-g\|_Y &\geqslant d_Y(f) \geqslant d_R(f)\\ &\geqslant \inf\{|(f-g)(y_i)| : i=1, ..., n+1\} = \|f-g\|_Y,\end{aligned}$$

whence $g \in P_Y(f)$.

THEOREM. 1. *Every function in $C(X) \sim G$ has a σ-alternator in G.*

2. *Every function in $C(X) \sim G$ has a unique σ-alternator in G iff any non-zero function in G has at most n distinct zeros.*

3. *If every function $f \in C(X) \sim G$ has a unique σ-alternator g_f in G, then the mapping S: $C(X) \to G$ defined by*

$$Sf = \begin{cases} g_f & \text{if } f \in C(X) \sim G \\ f & \text{if } f \in G \end{cases}$$

is a continuous selection of the metric projection of G.

Proof. 1. Fix $f \in C(X) \sim G$.

There exists a sequence $\{U_k\}_{k \in \mathbf{N}}$ in $\mathscr{U}$ such that

$$\lim_{k \in \mathbf{N}} \Omega(f, g_1, ..., g_n; U_k) = 0.$$

By the implication "(a) $\Rightarrow$ (c″)" in Lemma 1, for every $k \in \mathbf{N}$ there exists a finite U_k-net Y_k in X such that $G \mid Y_k$ is n-dimensional and satisfies the Haar condition. Set $P_{Y_k}(f) = \{h_k\}$ for every $k \in \mathbf{N}$.

S. I. Zuhovitzky [12] proved (the unordered alternation theorem for approximation by Chebyshev subspaces) that for each $k \in \mathbf{N}$ there exist a reference $R_k = (y_{1,k}, ..., y_{n+1,k})$ in Y_k and a sign $s_k \in \{-1, 1\}$ such that

$$(f-h_k)(y_{i,k}) = s_k(-1)^i \sigma_{R_k, i} \|f-h_k\|_{Y_k} \qquad \text{for} \quad i=1, ..., n+1.$$

By the first discretization lemma in the Appendix, the sequence $\{d_{Y_k}(f)\}_{k \in \mathbf{N}}$ converges to $d(f)$ and the sequence $\{h_k\}_{k \in \mathbf{N}}$ is a bounded sequence all of whose cluster points lie in $P(f)$. Let h be one of these cluster points. There exists a subnet $\{h_{k_l}\}_{l \in L}$ of the sequence $\{h_k\}_{k \in \mathbf{N}}$ which converges to h and for which

- for each $i=1, ..., n+1$, there exists a point $y_i \in X$ such that $\lim_{l \in L} y_{i,k_l} = y_i$;
- for each $i=1, ..., n+1$, there exists a sign $s_i' \in \{-1, 1\}$ such that $\sigma_{R_{k_l}, i} = s_i'$ for all $l \in L$; and
- there exists a sign $s \in \{-1, 1\}$ such that $s_{k_l} = s$ for all $l \in L$.

Clearly

$$(f-h)(y_i)=\lim_{l\in L}(f-h_{k_l})(y_{i,k_l})=s(-1)^i s_i' \|f-h\|$$

$$\text{for} \quad i=1, ..., n+1.$$

Thus, if all the y_i are distinct, then $R=(y_1, ..., y_{n+1})$ is a reference in X and, by the Corollary, $s_i'=\sigma_{R,i}$ for $i=1, ..., n+1$, whence h is a σ-alternator of f in G. It remains to be seen why no two of the y_i can coincide.

Suppose that at least two of the y_i coincide. Choose auxiliary points if necessary to obtain distinct points $z_1, ..., z_n$ of X so that each y_i is a z_j, and use the Corollary to obtain disjoint open neighborhoods $U_1, ..., U_n$ of $z_1, ..., z_n$, respectively, and a sign $s'\in\{-1, 1\}$ such that

$$s'D(p)\geqslant 0 \qquad \text{for all} \quad p\in\prod_{i=1}^{n} U_i.$$

Fix $l\in L$ sufficiently large that

$$\text{for every} \quad i=1, ..., n+1, \qquad \text{if} \quad y_i\in U_j \qquad \text{then} \quad y_{i,k_l}\in U_j.$$

Obviously, there exist $\alpha_1, ..., \alpha_{n+1}\in\mathbf{R}$ not all zero such that $\sum_{i=1}^{n+1}\alpha_i v(y_{i,k_l})=0$. By the third fact about $\mathbf{R}^n$ in the Appendix, there exists a $\gamma\in\mathbf{R}\sim\{0\}$ such that

$$\alpha_i=\gamma(-1)^i D_{R_{k_l},i} \qquad \text{for} \quad i=1, ..., n+1.$$

Thus, all the α_i are non-zero and

$$\operatorname{sgn}\alpha_i=\operatorname{sgn}\gamma(-1)^i\sigma_{R_{k_l},i} \qquad \text{for} \quad i=1, ..., n+1.$$

Now, by our choice of $U_1, ..., U_n$, if y_{i,k_l}, $y_{j,k_l}\in U_k$, then $y_i=y_j$, whence

$$\begin{aligned} s(-1)^i s_i' \|f-h\| &= (f-h)(y_i)=(f-h)(y_j) \\ &= s(-1)^j s_j' \|f-h\|, \end{aligned}$$

whence $(-1)^i s_i'=(-1)^j s_j'$, whence $\operatorname{sgn}\alpha_i=\operatorname{sgn}\alpha_j$: We have reached a contradiction to Lemma 2.

2. Suppose first that some non-zero function g_0 in G has $n+1$ distinct zeros $x_1, ..., x_{n+1}$. Set $R=(x_1, ..., x_{n+1})$, choose a function $h\in C(X)$ with the properties

$$\|h\|=1 \qquad \text{and} \qquad h(x_i)=(-1)^i\sigma_{R,i} \qquad \text{for} \quad i=1, ..., n+1,$$

and set

$$f = h\left(1 - \frac{|g_0|}{\|g_0\|}\right).$$

Then $(-1)^i \sigma_{R,i} f(x_i) = 1$ for $i = 1, ..., n+1$, and therefore, by the de la Vallée Poussin estimates,

$$d(f) \geqslant d_R(f) \geqslant 1.$$

On the other hand, for $|c| \leqslant 1$,

$$\left|f - c\frac{g_0}{\|g_0\|}\right| \leqslant |f| + |c|\frac{|g_0|}{\|g_0\|} = |h|\left|1 - \frac{|g_0|}{\|g_0\|}\right| + |c|\frac{|g_0|}{\|g_0\|}$$
$$\leqslant \left(1 - \frac{|g_0|}{\|g_0\|}\right) + |c|\frac{|g_0|}{\|g_0\|} = 1 - (1 - |c|)\frac{|g_0|}{\|g_0\|} \leqslant 1.$$

Thus, $d(f) = 1$ and $c(g_0/\|g_0\|) \in P(f)$ for all $|c| \leqslant 1$. Now,

$$\left(f - c\frac{g_0}{\|g_0\|}\right)(x_i) = (-1)^i \sigma_{R,i} \qquad \text{for} \quad i = 1, ..., n+1 \quad \text{and} \quad |c| \leqslant 1$$

shows that all the $c(g_0/\|g_0\|)$, $|c| \leqslant 1$, are σ-alternators for f in G. So much for this half of 2.

In order to prove the other half of 2, we suppose that card $Z(g) \leqslant n$ for all $g \in G \sim \{0\}$ and recall that this means just that any reference in X is admissible. We commence with a

LEMMA. *If g is any function in G and if $R = (x_1, ..., x_{n+1})$ is a reference in X such that*

$$(-1)^i \sigma_{R,i}\, g(x_i) \leqslant 0 \qquad \textit{for} \quad i = 1, ..., n+1,$$

then, for every $i = 1, ..., n+1$ such that $D_{R,i} \neq 0$, the function $(-1)^i \sigma_{R,i}\, g$ is non-negative in a neighborhood of x_i.

Proof. Let g and R be as specified. Set

$$I = \{i \in \{1, ..., n+1\} : D_{R,i} \neq 0\}$$

and observe that

$$0 = \sum_{i=1}^{n+1} \mu_{R,i}\, g(x_i) = -\sum_{i=1}^{n+1} |\mu_{R,i}|\, |g(x_i)|,$$

whence

$$g(x_i)=0 \qquad \text{for all} \quad i\in I. \tag{$*$}$$

Set

$$J=\{1, ..., n+1\}\sim I.$$

If $J=\phi$ then, by $(*)$, $g=0$, so that the conclusion of the lemma holds for trivial reasons. Suppose therefore that $J\neq\phi$, set

$$H=\{h\in G : h(x_i)=0 \text{ for all } i\in I\},$$

and fix $i\in I$. By the implication "(a) $\Rightarrow$ (c)" in the fact from linear algebra in the Appendix,

$$\dim H=n-\operatorname{card} I+1=\operatorname{card} J.$$

For each $j\in J$, define a function $h_j\in G$ by

$$h_j(x)=\det(v(x_1), ..., \widehat{v(x_j)}, ..., v(x_{n+1}))_{x_i=x},$$

where the subscript "$x_i=x$" indicates that the point x_i in the determinant is to be replaced by the variable $x\in X$. Since

$$h_j(x_k)=\begin{cases} 0 & \text{if} \quad k\in\{1, ..., n+1\}\sim\{i, j\}, \\ D_{R,j}=0 & \text{if} \quad k=i, \\ (-1)^{i+j+1}D_{R,i}\neq 0 & \text{if} \quad k=j, \end{cases} \qquad j\in J,$$

the functions $\{h_j : j\in J\}$ form a basis for H. By $(*)$, $g\in H$, i.e.,

$$g=\sum_{j\in J} c_j h_j \qquad \text{for some} \quad c_j\in\mathbf{R}.$$

Now, since

$$\begin{aligned} -(-1)^j\sigma_{R,j}\,|g(x_j)| &= g(x_j)=c_jh_j(x_j) \\ &= c_j(-1)^{i+j+1}\sigma_{R,i}\,|D_{R,i}|, \qquad j\in J, \end{aligned}$$

we have that

$$(-1)^i\sigma_{R,i}\sigma_{R,j}c_j\geqslant 0 \qquad \text{for all} \quad j\in J;$$

and by the Corollary, for each $j\in J$ there exists a neighborhood U_j of x_i such that

$$\sigma_{R,j}h_j(x)\geqslant 0 \qquad \text{for all} \quad x\in U_j.$$

Combining the last two sets of inequalities, we obtain that

$$(-1)^i \sigma_{R,i} g(x) = \sum_{j \in J} (-1)^i \sigma_{R,i} c_j h_j(x) = \sum_{j \in J} |c_j| \, |h_j(x)| \geqslant 0$$

for all $x \in \bigcap_{j \in J} U_j$. The lemma is proved.

We now suppose that some $f \in C(X) \sim G$ has two σ-alternators h_1 and h_2 in G, say,

$$(f - h_k)(x_{i,k}) = s_k(-1)^i \sigma_{R_k,i} \|f - h_k\|, \qquad i = 1, ..., n+1, \quad k = 1, 2,$$

for references $R_k = (x_{1,k}, ..., x_{n+1,k})$ in X and signs $s_k \in \{-1, 1\}$. Set

$$X_k = \{x_{i,k} : D_{R_k,i} \neq 0\}, \qquad k = 1, 2.$$

For any $g \in P(f)$,

$$\begin{aligned} &s_k(-1)^i \sigma_{R_k,i}(h_k - g)(x_{i,k}) \\ &\quad = s_k(-1)^i \sigma_{R_k,i}(f - g)(x_{i,k}) - s_k(-1)^i \sigma_{R_k,i}(f - h_k)(x_{i,k}) \\ &\quad = s_k(-1)^i \sigma_{R_k,,i}(f - g)(x_{i,k}) - d(f) \leqslant 0, \\ &\qquad i = 1, ..., n+1, \qquad k = 1, 2. \end{aligned}$$

A first two-fold appeal to the lemma, once with $h_1 - h_2$ on R_1 and once with $h_2 - h_1$ on R_2, tells us that $h_1 = h_2$ on $X_1 \cup X_2$. Thus if $\operatorname{card}(X_1 \cup X_2) \geqslant n+1$, $h_1 = h_2$, and we are done. Suppose therefore that $\operatorname{card}(X_1 \cup X_2) \leqslant n$. Set

$$H_k = \{h \in G : h = 0 \text{ on } X_k\}, \qquad k = 1, 2.$$

Since

$$H_1 \cap H_2 = \{h \in G : h = 0 \text{ on } X_1 \cup X_2\}$$

and

$$H_1 + H_2 \subset \{h \in G : h = 0 \text{ on } X_1 \cap X_2\},$$

by the implication "(a) $\Rightarrow$ (c)" in the fact from linear algebra in the Appendix,

$$\begin{aligned} n - \operatorname{card}(X_1 \cap X_2) + 1 &\geqslant \dim\{h \in G : h = 0 \text{ on } X_1 \cap X_2\} \\ &\geqslant \dim(H_1 + H_2) = \dim H_1 + \dim H_2 - \dim(H_1 \cap H_2) \\ &= (n - \operatorname{card} X_1 + 1) + (n - \operatorname{card} X_2 + 1) \\ &\quad - (n - \operatorname{card}(X_1 \cup X_2) + 1) = n - \operatorname{card}(X_1 \cap X_2) + 1, \end{aligned}$$

whence

$$\dim\{h \in G : h = 0 \text{ on } X_1 \cap X_2\} = n - \operatorname{card}(X_1 \cap X_2) + 1.$$

This implies first that $X_1 \cap X_2 \neq \phi$ and then that the vectors $\{v(x)\}_{x \in X_1 \cap X_2}$ are linearly dependent. By the implication "(a) $\Rightarrow$ (d)" in the fact from linear algebra in the Appendix, the latter is possible only if $X_1 = X_2$. Since $h_1 = h_2$ on $X_1 = X_2$, for any $x_{i,1} \in X_1$ and any $x_{j,2} \in X_2$ such that $x_{i,1} = x_{j,2}$,

$$s_1(-1)^i \sigma_{R_1,i} d(f) = (f - h_1)(x_{i,1}) = (f - h_2)(x_{j,2}) = s_2(-1)^j \sigma_{R_2,j} d(f),$$

whence

$$s_1(-1)^i \sigma_{R_1,i} = s_2(-1)^j \sigma_{R_2,j}.$$

Now, a second two-fold appeal to the lemma tells us that

$$h_1 = h_2 \qquad \text{in a neighborhood of } X_1 = X_2,$$

and this, since not all points of $X_1 = X_2$ are isolated points of X, because G is an almost Chebyshev subspace of $C(X)$, finally implies that $h_1 = h_2$ also in this case.

3. Suppose that every function $f \in C(X) \sim G$ has a unique σ- alternator g_f in G and suppose that the selection S of the metric projection P of G has been defined according to 3. Since P is upper semi-continuous, S is continuous at all points of G. Suppose therefore that $\{f_k\}_{k \in \mathbf{N}}$ is a sequence in $C(X) \sim G$ which converges to $f \in C(X) \sim G$. For every $k \in \mathbf{N}$, let $R_k = (x_{1,k}, ..., x_{n+1,k})$ be a reference in X and $s_k \in \{-1, 1\}$ a sign such that

$$(f_k - Sf_k)(x_{i,k}) = s_k(-1)^i \sigma_{R_k,i} \|f_k - Sf_k\| \qquad \text{for} \quad i = 1, ..., n+1.$$

The sequence $\{Sf_k\}_{k \in \mathbf{N}}$ is a bounded sequence all of whose cluster points lie in $P(f)$. Let g be one of these cluster points. There exists a subnet $\{Sf_{k_l}\}_{l \in L}$ of the sequence $\{Sf_k\}_{k \in \mathbf{N}}$ which converges to g and for which

- for each $i = 1, ..., n+1$, there exists a point $x_i \in X$ such that $\lim_{l \in L} x_{i,k_l} = x_i$;
- for each $i = 1, ..., n+1$, there exists a sign $s_i' \in \{-1, 1\}$ such that $\sigma_{R_{k_l},i} = s_i'$ for all $l \in L$; and
- there exists a sign $s \in \{-1, 1\}$ such that $s_{k_l} = s$ for all $l \in L$.

Clearly,

$$(f - g)(x_i) = \lim_{l \in L} (f_{k_l} - Sf_{k_l})(x_{i,k_l}) = s(-1)^i s_i' \|f - g\| \qquad \text{for} \quad i = 1, ..., n+1.$$

Suppose that at least two of the x_i coincide. Choose auxiliary points if necessary to obtain distinct points $z_1, ..., z_n$ of X so that each x_i is a z_j, and use the Corollary to obtain disjoint open neighborhoods $U_1, ..., U_n$ of $z_1, ..., z_n$, respectively, and a sign $s' \in \{-1, 1\}$ such that

$$s'D(p) \geqslant 0 \qquad \text{for all} \quad p \in \prod_{i=1}^{n} U_i.$$

Fix $l \in L$ sufficiently large that

$$\text{for every} \quad i = 1, ..., n+1, \qquad \text{if} \quad x_i \in U_j \qquad \text{then} \quad x_{i,k_l} \in U_j.$$

By the Corollary, there exist disjoint open neighborhoods $V_1, ..., V_{n+1}$ of $x_{1,k_l}, ..., x_{n+1,k_l}$, respectively, such that

$$\text{for every} \quad i = 1, ..., n+1, s_i' D(p) \geqslant 0$$
$$\text{for all} \quad p \in V_1 \times \cdots \times \widehat{V_i} \times \cdots \times V_{n+1}$$

and

$$\text{for every} \quad i = 1, ..., n+1, \qquad \text{if} \quad x_{i,k_l} \in U_j \qquad \text{then} \quad V_i \subset U_j.$$

Use the implication "(a) $\Rightarrow$ (c′)" in Lemma 1 to obtain points $x_i^* \in V_i$, $i = 1, ..., n+1$, such that $G | \{x_1^*, ..., x_{n+1}^*\}$ is n-dimensional and satisfies the Haar condition, and set $R^* = (x_1^*, ..., x_{n+1}^*)$. Obviously, there exist $\alpha_1, ..., \alpha_{n+1} \in \mathbf{R}$ not all zero such that $\sum_{i=1}^{n+1} \alpha_i v(x_i^*) = 0$. By the third fact about $\mathbf{R}^n$ in the Appendix, there exists a $\gamma \in \mathbf{R} \sim \{0\}$ such that

$$\alpha_i = \gamma(-1)^i D_{R^*,i} \qquad \text{for} \quad i = 1, ..., n+1.$$

Thus, all the α_i are non-zero and

$$\operatorname{sgn} \alpha_i = \operatorname{sgn} \gamma(-1)^i \sigma_{R^*,i} \qquad \text{for} \quad i = 1, ..., n+1.$$

Now, by our choice of $V_1, ..., V_{n+1}$,

$$\sigma_{R^*,i} = \sigma_{R_{k_l},i} = s_i' \qquad \text{for} \quad i = 1, ..., n+1,$$

and therefore, if $x_i^*, x_j^* \in U_k$, then $x_i = x_j$, whence

$$\begin{aligned} s(-1)^i s_i' \, \|f-h\| &= (f-h)(x_i) = (f-h)(x_j) \\ &= s(-1)^j s_j' \, \|f-h\|, \end{aligned}$$

whence $(-1)^i s_i' = (-1)^j s_j'$, whence $\operatorname{sgn} \alpha_i = \operatorname{sgn} \alpha_j$: We have reached a contradiction to Lemma 2. This shows that no two of the x_i coincide. Thus $R = (x_1, ..., x_{n+1})$ is a reference in X and, by the Corollary, $\sigma_{R,i} = s_i'$ for

$i=1, \ldots, n+1$, whence $g=Sf$. This shows that $\lim_{k \in \mathbf{N}} Sf_k = Sf$ and we are done.

Remarks. 1. The Theorem, of course, characterizes the values of the unique continuous selection in Blatter's theorem as unique σ-alternators, just as the alternation theorem characterizes unique best approximations as unique alternators. We call attention to the fact, however, that our proof of the Theorem also provides a new and simpler proof of the difficult half of Blatter's theorem; that we were working on such a proof was announced in [1].

2. We note that the function f used in the first part of the proof of 2 is just the function Haar used in the proof of his theorem.

3. Simple examples show that the σ-alternators in 1 cannot always be taken on admissible references. Here is one: Set $X=[-1, 1]$, define $g_1 \in C(X)$ by

$$g_1(x)=(1-|x|)^{1/2}, \qquad |x| \leqslant 1,$$

and set $G=\operatorname{span}\{g_1\}$. Then G is a 1-dimensional weakly interpolating almost Chebyshev subspace of $C(X)$, the function $f \in C(X) \sim G$ defined by

$$f(x)=x, \qquad |x| \leqslant 1,$$

has 0 for its only best approximation in G, and the only reference on which f σ-alternates is the non-admissible reference $R=(-1, 1)$.

4. A. J. Lazar, P. D. Morris, and D. E. Wulbert [8] proved the following

THEOREM. *If G is 1-dimensional, its metric projection has a continuous selection iff*

(i) card bdry $Z(g_1) \leqslant 1$ (bdry = boundary of); *and*

(ii) *if* bdry $Z(g_1)=\{x\}$, *then one of g_1 and $-g_1$ is non-negative in a neighborhood of x.*

In order to prove the sufficiency part of their theorem, Lazar, Morris, and Wulbert set

$$H=\{g \mid X \sim \operatorname{int} Z(g_1) : g \in G\}$$

and observe that, given a continuous selection S' of the metric projection of $C(X \sim \operatorname{int} Z(g_1))$ onto H, the mapping $S: C(X) \rightarrow G$ defined by

$$Sf(x)=\begin{cases} S'(f \mid X \sim \operatorname{int} Z(g_1))(x) & \text{if } \ x \in X \sim \operatorname{int} Z(g_1), \\ 0 & \text{if } \ x \in \operatorname{int} Z(g_1), \end{cases} \qquad f \in C(X),$$

is a continuous selection of the metric projection of G; they then set out to construct such an S'. Now, it is obvious that H is a 1-dimensional weakly interpolating almost Chebyshev subspace of $C(X \sim \operatorname{int} Z(g_1))$ with the property that card $Z(h) \leqslant 1$ for all $h \in H \sim \{0\}$, and therefore, by our Theorem, the metric projection of H has a unique continuous selection, namely, the mapping which leaves the elements of H fixed and sends each element of $C(X \sim \operatorname{int} Z(g_1)) \sim H$ onto its unique σ-alternator in H. Thus, the Lazar–Morris–Wulbert selection may be obtained via unique σ-alternators.

5. M. Sommer [10, 11], in his approach to σ-alternators, uses, among others, as a crucial condition on G that it be Haar on the complement of some finite subset of X. We remark in passing that J. Blatter [1], in his extension of Mairhuber's theorem, has provided examples of G's which admit unique σ-alternatores but do not satisfy this condition.

3. Calculating Unique Continuous Selections

For this section we assume that G is a weakly interpolating almost Chebyshev subspace of $C(X)$ with the property that any non-zero function in G has at most n distinct zeros, and that f is a fixed function in $C(X)$ which does not belong to G. We want to design an iterative algorithm which calculates the value at f of the unique continuous selection of the metric projection of G, that is to say, the unique σ-alternator of f in G.

The basic process in this algorithm is that of solving systems of linear equations with a matrix

$$M_R = \begin{bmatrix} g_1(x_1) & \cdots & g_n(x_1) & (-1)^1 \sigma_{R,1} \\ \vdots & & \vdots & \vdots \\ g_1(x_{n+1}) & \cdots & g_n(x_{n+1}) & (-1)^{n+1} \sigma_{R,n+1} \end{bmatrix},$$

$R = (x_1, ..., x_{n+1})$ a reference in X. The Laplace development of det M_R by the last column is

$$\det M_R = \sum_{i=1}^{n+1} (-1)^{n+1+i} (-1)^i \sigma_{R,i} D_{R,i}.$$

Since $\sigma_{R,i} D_{R,i} \geqslant 0$ for all i, and since $D_{R,i} \neq 0$ for some i, it follows that

$$(-1)^{n+1} \det M_R > 0;$$

i.e., we are dealing with non-singular systems. Exactly which systems we are solving, and why, is explained in

Approximation on a Reference. *Let $R = (x_1, ..., x_{n+1})$ be a reference* *in X with the property that $d_R(f) > 0$ (since $\dim(G + \mathbf{R}f) = n+1$, such references exist!). Then the solution $(c_{R,1}, ..., c_{R,n+1}) \in \mathbf{R}^{n+1}$ of the system*

$$M_R \begin{bmatrix} c_{R,1} \\ \vdots \\ c_{R,n+1} \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_{n+1}) \end{bmatrix}$$

has the properties

(1) *the function $g_R = \sum_{i=1}^{n} c_{R,i} g_i$ is a σ-alternator on R of f in G;*

(2) *the modulus $d_R = |c_{R,n+1}|$ is the distance on R of f to G; and*

(3) *the sign $s_R = \operatorname{sgn} c_{R,n+1}$ satisfies the identity*

$$s_R d_R = \sum_{i=1}^{n+1} \mu_{R,i} f(x_i).$$

Proof. By the definitions involved,

$$(f - g_R)(x_i) = (-1)^i \sigma_{R,i} c_{R,n+1} \qquad \text{for} \quad i = 1, ..., n+1.$$

This shows first that $c_{R,n+1} \neq 0$ ($d_R(f) > 0$!), and then that g_R is a σ-alternator on R of f in G. Thus, by the de la Vallée Poussin estimates, $g_R \in P_R(f)$, and therefore $d_R = \|f - g_R\|_R = d_R(f)$. Finally,

$$\sum_{i=1}^{n+1} \mu_{R,i} f(x_i) = \sum_{i=1}^{n+1} \mu_{R,i} (f - g_R)(x_i) = s_R d_R \sum_{i=1}^{n+1} (-1)^i \sigma_{R,i} \mu_{R,i} = s_R d_R.$$

The key device in our algorithm is an exchange procedure E which assigns to each pair (R, x) in the set

> $\mathscr{R} = \{(R, x) : R$ is a reference in X with the property that $d_R(f) > 0$, and x is a point in X with the property that $|(f - g_R)(x)| > d_R$; so that, in particular, the point x does not belong to the reference $R\}$

an exchange reference $E(R, x)$ in X, namely, the reference R with one of its points exchanged for x; the exchange index $e(R, x)$, that is to say, the index of the point of R to be exchanged for x, is given by

THE EXCHANGE RULE. *Let $(R = (x_1, ..., x_{n+1}), x) \in \mathscr{R}$ and set $s = \operatorname{sgn}(f - g_R)(x)$. Then there exists a unique index $m = e(R, x)$ in $\{1, ..., n+1\}$ with the property that, if the reference $R' = (x'_1, ..., x'_{n+1}) = E(R, x)$ in X is defined by*

$$x'_i = \begin{cases} x & \text{if} \quad i = m \\ x_i & \text{if} \quad i \in \{1, ..., n+1\} \sim \{m\}, \end{cases}$$

then, for some $s' \in \{-1, 1\}$,

$$s'(-1)^i \sigma_{R',i}(f-g_R)(x_i') > 0 \qquad \textit{for} \quad i = 1, ..., n+1, \tag{1}$$

or, equivalently (note that $\sigma_{R',m} = \sigma_{R,m}$*!),*

$$\sigma_{R',i} = ss_R(-1)^m \sigma_{R,m} \sigma_{R,i} \qquad \textit{for} \quad i \in \{1, ..., n+1\} \sim \{m\}; \tag{1'}$$

this index m and the associated reference R' have the additional properties that, if $(v_{R,1}, ..., v_{R,n+1}) \in \mathbf{R}^{n+1}$ *is the solution of the system*

$$M_R^\tau \begin{bmatrix} v_{R,1} \\ \vdots \\ v_{R,n} \\ v_{R,n+1} \end{bmatrix} = \begin{bmatrix} ss_R\, g_1(x) \\ \vdots \\ ss_R\, g_n(x) \\ 1 \end{bmatrix} \qquad (^\tau = \text{transpose of}),$$

which is to say that

$$\sum_{i=1}^{n+1} v_{R,i} v(x_i) = ss_R v(x) \qquad \textit{and} \qquad \sum_{i=1}^{n+1} (-1)^i \sigma_{R,i} v_{R,i} = 1,$$

then

$$(-1)^m \sigma_{R,m} v_{R,m} > 0 \qquad \textit{and}$$

$$\frac{\mu_{R,m}}{v_{R,m}} = \inf \left\{ \frac{\mu_{R,i}}{v_{R,i}} : i \in \{1, ..., n+1\} \textit{ and } (-1)^i \sigma_{R,i} v_{R,i} > 0 \right\}; \tag{2}$$

$$\mu_{R',m} = (-1)^m \sigma_{R,m} \frac{\mu_{R,m}}{v_{R,m}} \qquad \textit{and}$$

$$\mu_{R',i} = ss_R(-1)^m \sigma_{R,m} \left(\mu_{R,i} - \frac{\mu_{R,m}}{v_{R,m}} v_{R,i} \right) \qquad \textit{for} \quad i \in \{1, ..., n+1\} \sim \{m\}; \tag{3}$$

$$d_{R'}(f) = d_R(f) + \frac{\mu_{R,m}}{v_{R,m}} (|(f-g_R)(x)| - d_R(f)). \tag{4}$$

Proof. Unicity. Suppose that two distinct indices m_1 and m_2 have the required property (1′), and denote by R_1' and R_2' the respective new references. Then, by (1′),

$$\sigma_{R_1',m_2} = ss_R(-1)^{m_1} \sigma_{R,m_1} \sigma_{R,m_2} \qquad \text{and} \qquad \sigma_{R_2',m_1} = ss_R(-1)^{m_2} \sigma_{R,m_2} \sigma_{R,m_1},$$

whence

$$\sigma_{R_2',m_1} = (-1)^{m_1+m_2} \sigma_{R_1',m_2},$$

whereas, by the definitions involved,

$$\sigma_{R_2',m_1} = -(-1)^{m_1+m_2}\sigma_{R_1',m_2}.$$

Existence. By the Corollary, by what we have seen in approximation on a reference, and by the de la Vallée Poussin estimates, there exist disjoint neighborhoods $U_1, .., U_{n+1}$ and U of $x_1, ..., x_{n+1}$ and x, respectively, with the properties

$$\sigma \text{ is constant on the product of any } n \text{ of } U_1, ..., U_{n+1} \text{ and } U; \tag{5}$$

for any reference $\mathrm{Ref} \in U_1 \times \cdots \times U_{n+1}$ and for any point $y \in U$,

$$d_{\mathrm{Ref}}(f) > 0, \qquad s_{\mathrm{Ref}} = s_R, \qquad s(f - g_{\mathrm{Ref}})(y) > d_{\mathrm{Ref}}. \tag{6}$$

Comparing this choice of $U_1, ..., U_{n+1}$ and U with condition (1′), we see that if the exchange index map e is to exist, we must have $e(\mathrm{Ref}, y) = e(R, x)$ for all pairs (Ref, y) with $\mathrm{Ref} \in U_1 \times \cdots \times U_{n+1}$ and $y \in U$. We use a special such pair to construct $e(R, x)$. By the implication "(a) ⇒ (c′)" in Lemma 1, there exist $x_1^*, ..., x_{n+1}^*$ and x^* in $U_1, ..., U_{n+1}$ and U, respectively, such that the restriction of G to $\{x_1^*, ..., x_{n+1}^*\} \cup \{x^*\}$ is n-dimensional and satisfies the Haar condition. We set $R^* = (x_1^*, ..., x_{n+1}^*)$ and we denote by $(v_{R^*,1}, ..., v_{R^*,n+1}) \in \mathbf{R}^{n+1}$ the solution of the system

$$M_{R^*}^{\tau} \begin{bmatrix} v_{R^*,1} \\ \vdots \\ v_{R^*,n} \\ v_{R^*,n+1} \end{bmatrix} = \begin{bmatrix} ss_{R^*}g_1(x^*) \\ \vdots \\ ss_{R^*}g_n(x^*) \\ 1 \end{bmatrix}.$$

Since $\sum_{i=1}^{n+1} (-1)^i \sigma_{R^*,i} v_{R^*,i} = 1$, the set

$$I = \{i \in \{1, ..., n+1\} : (-1)^i \sigma_{R^*,i} v_{R^*,i} > 0\}$$

is non-empty. We choose an index $m \in I$ with the property that

$$\frac{\mu_{R^*,m}}{v_{R^*,m}} = \inf\left\{\frac{\mu_{R^*,i}}{v_{R^*,i}} : i \in I\right\},$$

we define a reference $R^{*\prime} = (x_1^{*\prime}, ..., x_{n+1}^{*\prime})$ in X by

$$x_i^{*\prime} = \begin{cases} x^* & \text{if } \; i = m \\ x_i^* & \text{if } \; i \in \{1, ..., n+1\} \sim \{m\}, \end{cases}$$

and we claim that

$$\sigma_{R^{*\prime},i} = ss_{R^*}(-1)^m \sigma_{R^*,m}\sigma_{R^*,i} \qquad \text{for} \quad i \in \{1, ..., n+1\} \sim \{m\}. \tag{7}$$

In order to prove this claim, we set

$$\alpha_m = (-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \qquad \text{and}$$
$$\alpha_i = ss_{R^*}(-1)^m \sigma_{R^*,m} \left(\mu_{R^*,i} - \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} \right) \qquad \text{for} \quad i \in \{1, ..., n+1\} \sim \{m\}. \tag{8}$$

Then

$$\sum_{i=1}^{n+1} \alpha_i v(x_i^{*\prime})$$
$$= (-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} v(x^*) + \sum_{\substack{i=1 \\ i \neq m}}^{n+1} ss_{R^*}(-1)^m \sigma_{R^*,m}$$
$$\times \left(\mu_{R^*,i} - \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} \right) v(x_i^*)$$
$$= (-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} v(x^*) + \sum_{i=1}^{n+1} ss_{R^*}(-1)^m \sigma_{R^*,m}$$
$$\times \left(\mu_{R^*,i} - \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} \right) v(x_i^*)$$
$$= (-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} v(x^*) - \sum_{i=1}^{n+1} ss_{R^*}(-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} v(x_i^*)$$
$$= ss_{R^*}(-1)^m \sigma_{R^*,m} \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \left(ss_{R^*} v(x^*) - \sum_{i=1}^{n+1} \nu_{R^*,i} v(x_i^*) \right) = 0. \tag{9}$$

From observation of the fact that $(-1)^i \sigma_{R^*,i} \mu_{R^*,i} > 0$ for $i = 1, ..., n+1$, it follows from the definition of I and the choice of m that

$$ss_{R^*}(-1)^m \sigma_{R^*,m} (-1)^i \sigma_{R^*,i} \alpha_i \geqslant 0 \qquad \text{for} \quad i \in \{1, ..., n+1\} \sim \{m\}. \tag{10}$$

Using (10), we obtain from (8) that

$$\sum_{i=1}^{n+1} |\alpha_i| = \frac{\mu_{R^*,m}}{\nu_{R^*,m}} + \sum_{\substack{i=1 \\ i \neq m}}^{n+1} (-1)^i \sigma_{R^*,i} \left(\mu_{R^*,i} - \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} \right)$$
$$= \frac{\mu_{R^*,m}}{\nu_{R^*,m}} + \sum_{i=1}^{n+1} (-1)^i \sigma_{R^*,i} \left(\mu_{R^*,i} - \frac{\mu_{R^*,m}}{\nu_{R^*,m}} \nu_{R^*,i} \right) = 1. \tag{11}$$

Combining (9) and (11) with the third fact about $\mathbf{R}^n$ in the Appendix, we see that there exists a $\gamma \in \{-1, 1\}$ such that

$$\mu_{R^{*\prime},i} = \gamma \alpha_i \qquad \text{for} \quad i = 1, ..., n+1. \tag{12}$$

Since $(-1)^i \sigma_{R^{*\prime},i} \mu_{R^{*\prime},i} > 0$ for $i = 1, ..., n+1$ (and since $\sigma_{R^{*\prime},m} = \sigma_{R^*,m}$!), (12) and (8) imply that $\gamma = 1$, and then (12) and (10) imply (7). Now observe that (7), (6), and (5) imply (1′).

Additional properties. By the definitions involved,

$$\mu_{R',m} = \frac{(-1)^m D_{R',m}}{|\det M_{R'}|} = \frac{(-1)^m D_{R,m}}{|\det M_{R'}|} = \frac{|\det M_R|}{|\det M_{R'}|} \mu_{R,m}, \tag{13}$$

and, using the fact that $v(x) = ss_R \sum_{i=1}^{n+1} v_{R,i} v(x_i)$,

$$\begin{aligned} \mu_{R',i} &= \frac{(-1)^i}{|\det M_{R'}|} ss_R(v_{R,m} D_{R,i} + (-1)^{m+i+1} v_{R,i} D_{R,m}) \\ &= \frac{|\det M_R|}{|\det M_{R'}|} ss_R(\mu_{R,i} v_{R,m} - \mu_{R,m} v_{R,i}) \qquad \text{for} \quad i \in \{1, ..., n+1\} \sim \{m\}. \end{aligned} \tag{14}$$

Combining (13) and (14) with (1′), we see that

$$\begin{aligned} 1 &= \sum_{i=1}^{n+1} (-1)^i \sigma_{R',i} \mu_{R',i} \\ &= \frac{|\det M_R|}{|\det M_{R'}|} (-1)^m \sigma_{R,m} \left(\mu_{R,m} + \sum_{\substack{i=1 \\ i \neq m}}^{n+1} (-1)^i \sigma_{R,i} (\mu_{R,i} v_{R,m} - \mu_{R,m} v_{R,i}) \right) \\ &= \frac{|\det M_R|}{|\det M_{R'}|} (-1)^m \sigma_{R,m} v_{R,m}. \end{aligned} \tag{15}$$

Now, (15) implies immediately that $(-1)^m \sigma_{R,m} v_{R,m} > 0$, which is the first part of (2); plugging (15) into (13) and (14) gives (3); by (3) and (1′),

$$0 \leqslant (-1)^i \sigma_{R',i} \mu_{R',i} = (-1)^i \sigma_{R,i} \left(\mu_{R,i} - \frac{\mu_{R,m}}{v_{R,m}} v_{R,i} \right) \qquad \text{for} \quad i = 1, ..., n+1,$$

and this trivially implies the second part of (2); finally, by (1′),

$$(s(-1)^m \sigma_{R,m})(-1)^i \sigma_{R',i} (f - g_R)(x_i') > 0 \qquad \text{for} \quad i = 1, ..., n+1,$$

whence, by the de la Vallée Poussin estimates,

$$d_R(f) > 0 \qquad \text{and} \qquad s_{R'} = s(-1)^m \sigma_{R,m},$$

and it follows, by what we have seen in approximation on a reference and by (3), that

$$\begin{aligned} d_{R'}(f) = d_{R'} = s_{R'} \sum_{i=1}^{n+1} \mu_{R',i} f(x'_i) &= s_{R'} \sum_{i=1}^{n+1} \mu_{R',i} (f - g_R)(x'_i) \\ &= \sum_{i=1}^{n+1} (-1)^i \sigma_{R',i} \mu_{R',i} \, |(f-g_R)(x'_i)| \\ &= d_R(f) + \frac{\mu_{R,m}}{v_{R,m}} (|(f-g_R)(x)| - d_R(f)), \end{aligned}$$

and this is (4).

As to be expected, our algorithm commences with a discretization of our original problem. This discretization is solved by

THE DISCRETE ALGORITHM. *Let Y be a finite subset of X, and let R_1 be a reference in Y with the property that $d_{R_1}(f) > 0$. Then the algorithm*

1. *Set $R = R_1$.*
2. *Calculate g_R, d_R, and s_R.*
3. *Calculate a point $y \in Y$ with the property that $|(f-g_R)(y)| = \|f-g_R\|_Y$, and set $s = \operatorname{sgn}(f-g_R)(y)$.*
4. *Exhibit R, g_R, d_R s_R, y, $|(f-g_R)(y)|$, and s.*
5. *If $|(f-g_R)(y)| > d_R$, calculate $e(R, y)$ according to the exchange rule, set $R = E(R, y)$, and go to step 2.*
6. *If $|(f-g_R)(y)| = d_R$, stop.*

is finite, i.e., reaches step 6; it is obvious that when the algorithm reaches step 6, then g_R is a σ-alternator on Y of f in G.

Proof. Suppose that the discrete algorithm is not finite. It then exhibits, upon executing step 4, a sequence (R_1, y_1), (R_2, y_2), ... of pairs such that for $j = 1, 2, \ldots$

- *$R_j = (y_{1,j}, \ldots, y_{n+1,j})$ is a reference in Y with the property that $d_{R_j}(f) > 0$;*
- *y_j is a point in Y with the property that $|(f-g_{R_j})(y_j)| > d_{R_j}$; and*
- *$R_{j+1} = E(R_j, y_j)$.*

Since Y is finite,

$$R_{j_2} = R_{j_1} \qquad \text{for some} \quad 1 \leqslant j_1 < j_2.$$

Since, by (4) of the exchange rule,

$$d_{R_{j_1}}(f) \leqslant d_{R_{j_1+1}}(f) \leqslant \cdots \leqslant d_{R_{j_2}}(f),$$

it follows that all these numbers are equal to some d. Set

$$r = \inf\{|(f-g_{R_j})(y_j)| : j_1 \leqslant j \leqslant j_2\} > d,$$

and set

$$s_j = \operatorname{sgn}(f-g_{R_j})(y_j) \qquad \text{for} \quad j_1 \leqslant j \leqslant j_2.$$

By the Corollary, by what we have seen in approximation on a reference, and by the de la Vallée Poussin estimates, for each $j_1 \leqslant j \leqslant j_2$ there exist disjoint neighborhoods $V_{1,j}, ..., V_{n+1,j}$ and V_j of $y_{1,j}, ..., y_{n+1,j}$ and y_j, respectively, with the properties

- σ is constant on the product of any n of $V_{1,j}, ..., V_{n+1,j}$ and V_j; and
- for any reference $\text{Ref} \in V_{1,j} \times \cdots \times V_{n+1,j}$ and for any point $y \in V_j$, $0 < d_{\text{Ref}}(f) < \frac{1}{2}(d+r)$, $s_{\text{Ref}} = s_{R_j}$, and $|(f-g_{\text{Ref}})(y)| > \frac{1}{2}(d+r)$, whence, by the exchange rule, $e(\text{Ref}, y) = e(R_j, y_j)$.

Set

$$\{x_1, ..., x_N\} = \bigcup_{j_1 \leqslant j \leqslant j_2} \{y_{1,j}, ..., y_{n+1,j}\} \cup \{y_j\},$$

define a function φ: $\{1, ..., n+1\} \times \{j_1, ..., j_2\} \to \{1, ..., N\}$ by

$$\text{if} \quad y_{i,j} = x_k, \qquad \text{then} \quad \varphi(i,j) = k,$$

define a function ψ: $\{j_1, ..., j_2\} \to \{1, ..., N\}$ by

$$\text{if} \quad y_j = x_k, \qquad \text{then} \quad \psi(j) = k,$$

and choose disjoint neighborhoods $U_1, ..., U_N$ of $x_1, ..., x_N$, respectively, such that

$$V_{i,j} \subset U_k \text{ whenever } \varphi(i,j) = k \qquad \text{and} \qquad V_j \subset U_k \text{ whenever } \psi(j) = k.$$

By the implication "(a) $\Rightarrow$ (c′)" in Lemma 1, there exist $x_1^*, ..., x_N^*$ in $U_1, ..., U_N$, respectively, such that $G \mid \{x_1^*, ..., x_N^*\}$ is n-dimensional and satisfies the Haar condition. Set

$$R_j^* = (x_{\varphi(1,j)}^*, ..., x_{\varphi(n+1,j)}^*) \qquad \text{for} \quad j_1 \leqslant j \leqslant j_2.$$

It is clear, then, that

$$R_{j+1}^* = E(R_j^*, x_{\psi(j)}^*) \qquad \text{for} \quad j_1 \leqslant j \leqslant j_2.$$

Since $\mu_{R_j^*, e(R_j^*, x_{\psi(j)}^*)} \neq 0$ for $j_1 \leqslant j \leqslant j_2$, it follows, by (4) of the exchange rule, that

$$d_{R_{j_1}^*}(f) < d_{R_{j_1+1}^*}(f) < \cdots < d_{R_{j_2}^*}(f),$$

whence, in particular,

$$R_{j_1}^* \neq R_{j_2}^*.$$

We have reached a contradiction.

THE ALGORITHM. *Let Y_1 be a finite subset of X and let R_1 be a reference in Y_1 with the property that $d_{R_1}(f) > 0$. Then the algorithm*

1. *Set $Y = Y_1$ and $R = R_1$.*
2. *Calculate g_R, d_R, and s_R.*
3. *Calculate a point $x \in Y$ with the property that $|(f-g_R)(x)| = \|f-g_R\|_Y$.*
4. *If $|(f-g_R)(x)| > d_R$, calculate $e(R, x)$ according to the exchange rule, set $R = E(R, x)$, and go to step 2.*
5. *If $|(f-g_R)(x)| = d_R$, calculate a point $x \in X$ with the property that $|(f-g_R)(x)| = \|f-g_R\|$ and set $s = \operatorname{sgn}(f-g_R)(x)$.*
6. *Exhibit Y, R, g_R, d_R, s_R, x, $|(f-g_R)(x)|$, and s.*
7. *If $|(f-g_R)(x)| > d_R$, calculate $e(R, x)$ according to the exchange rule, set $R = E(R, x)$, set $Y = Y \cup \{x\}$, and go to step 2.*
8. *If $|(f-g_R)(x)| = d_R$, stop.*

either is finite, i.e., reaches step 8, or else is not; in the former case, it is obvious that when the algorithm reaches step 8, then g_R is the σ-alternator of f in G; in the latter case, the algorithm produces a sequence of functions in G which converges to the σ-alternator of f in G.

Proof. Suppose that the algorithm is not finite. As we have seen in the discrete algorithm, it then exhibits, upon executing step 6, an increasing sequence $\{Y_k\}_{k \in \mathbf{N}}$ of finite subsets of X such that $\dim(G | Y_1) = n$, and for each Y_k a reference $R_k = (x_{1,k}, ..., x_{n+1,k})$ in Y_k with the property that g_{R_k} is a σ-alternator on Y_k of f in G. By the de la Vallée Poussin estimates, $g_{R_k} \in P_{Y_k}(f)$ for $k = 1, 2, ...$, and thus, by the second discretization lemma in the Appendix, the sequence $\{d_{Y_k}(f)\}_{k \in \mathbf{N}}$ converges to $d(f)$, and the sequence $\{g_{R_k}\}_{k \in \mathbf{N}}$ is a bounded sequence all of whose cluster points lie in $P(f)$. Let g be one of these cluster points. There exists a subnet $\{g_{R_{k_l}}\}_{l \in L}$ of the sequence $\{g_{R_k}\}_{k \in \mathbf{N}}$ which converges to g and for which

- for each $i = 1, ..., n+1$, there exists a point $x_i \in X$ such that $\lim_{l \in L} x_{i,k_l} = x_i$;

• for each $i=1, \ldots, n+1$, there exists a sign $s_i \in \{-1, 1\}$ such that $\sigma_{R_{k_l}, i} = s_i$ for all $l \in L$; and

• there exists a sign $s \in \{-1, 1\}$ such that $s_{R_{k_l}} = s$ for all $l \in L$.

Clearly,

$$(f-g)(x_i) = \lim_{l \in L} (f - g_{R_{k_l}})(x_{i,k_l}) = s(-1)^i s_i \|f-g\| \qquad \text{for} \quad i=1, \ldots, n+1.$$

Just as in the final paragraph of the proof of 3 in the Theorem, one sees that no two of the x_i coincide. Thus, $R=(x_1, \ldots, x_{n+1})$ is a reference in X and, by the Corollary, $\sigma_{R,i} = s_i$ for $i=1, \ldots, n+1$, whence g is the σ-alternator of f in G. This shows that the sequence $\{g_{R_k}\}_{k \in \mathbf{N}}$ converges to the σ-alternator of f in G, and we are done.

Remarks. 1. If, in the situation of the exchange rule, all the $\mu_{R',i}$ are non-zero, then the form of the $\mu_{R',i}$ given in (3) shows that condition (2) actually characterizes the exchange index m, i.e., the inf in (2) is attained only at m; this, unfortunately, is not true in general, not even if all of the $\mu_{R,i}$ are non-zero.

2. The σ-alternators on finite subsets of X of f in G which we calculate in the discrete algorithm are actually unique: repeat, verbatim, our proof of the corresponding part of the Theorem.

3. There is a very short, very elegant, but highly non-algorithmic proof for the existence of σ-alternators on finite subsets of X of f in G: use the implication "(a) $\Rightarrow$ (c′)" in Lemma 1.

APPENDIX

THREE FACTS ABOUT $\mathbf{R}^n$. 1. *Given a non-empty subset A of $\mathbf{R}^n$,*

$$0 \in \operatorname{int} \operatorname{conv}(A) \qquad (\text{conv} = \text{convex hull of})$$

iff there exist $N \geqslant n+1$ distinct points $a_1, \ldots, a_N \in A$ which span $\mathbf{R}^n$ and positive real numbers $\alpha_1, \ldots, \alpha_N$ such that $\sum_{i=1}^N \alpha_i a_i = 0$.

2. *Given $N \geqslant n+1$ distinct points $a_1, \ldots, a_N \in \mathbf{R}^n$ which span $\mathbf{R}^n$ and positive real numbers $\alpha_1, \ldots, \alpha_N$ such that $\sum_{i=1}^N \alpha_i a_i = 0$, then any N points $b_1, \ldots, b_N$ sufficiently close to $a_1, \ldots, a_N$, respectively, are also distinct and span $\mathbf{R}^n$ and have the property that $\sum_{i=1}^N \beta_i b_i = 0$ for some positive real numbers $\beta_1, \ldots, \beta_N$.*

3. *Given $n+1$ distinct points $a_1, \ldots, a_{n+1} \in \mathbf{R}^n$ which span $\mathbf{R}^n$ and real numbers $\alpha_1, \ldots, \alpha_{n+1}$ not all zero such that $\sum_{i=1}^{n+1} \alpha_i a_i = 0$, there exists a $\gamma \in \mathbf{R} \sim \{0\}$ such that*

$$\alpha_i = \gamma(-1)^i \det(a_1, \ldots, \widehat{a_i}, \ldots, a_{n+1}) \qquad \textit{for} \quad i=1, \ldots, n+1.$$

Proof. 1. Suppose first that $0 \in \operatorname{int}\operatorname{conv}(A)$. Let $b_1, ..., b_n$ be a basis for $\mathbf{R}^n$ and let $\varepsilon > 0$ be small enough that $\varepsilon b_1, ..., \varepsilon b_n, \ -\varepsilon \sum_{i=1}^n b_i \in \operatorname{conv}(A)$. Obviously,

$$\frac{1}{n+1}\sum_{j=1}^{n} \varepsilon b_j + \frac{1}{n+1}\left(-\varepsilon \sum_{i=1}^{n} b_i\right) = 0.$$

Now write $\varepsilon b_1, ..., \varepsilon b_n, \ -\varepsilon \sum_{i=1}^n b_i$ as convex combinations of elements of A to obtain the desired $a_1, ..., a_N$ and $\alpha_1, ..., \alpha_N$.

Now suppose that $N \geqslant n+1$ distinct points $a_1, ..., a_N \in A$ span $\mathbf{R}^n$ and have the property that $\sum_{i=1}^N \alpha_i a_i = 0$ for some positive real numbers $\alpha_1, ..., \alpha_N$. Since $a_1, ..., a_N$ span $\mathbf{R}^n$, the linear map

$$\mathbf{R}^N \to \mathbf{R}^n$$

$$(\beta_1, ..., \beta_N) \mapsto \sum_{i=1}^{N} \beta_i a_i$$

is onto and thus open. Now observe that

$$B = \left\{(\beta_1, ..., \beta_N) \in \mathbf{R}^N : \beta_1, ..., \beta_N > 0 \text{ and } \sum_{i=1}^{N} \beta_i < 1\right\}$$

is an open subset of $\mathbf{R}^N$ whose image under this map contains 0 and is contained in $\operatorname{conv}(A)$.

2. We may and shall assume that $a_1, ..., a_n$ are linearly independent. Then for any $b_1, ..., b_N \in \mathbf{R}^n$ sufficiently close to $a_1, ..., a_N$, respectively, $b_1, ..., b_n$ are also linearly independent, and the solution $(\beta_1, ..., \beta_n) \in \mathbf{R}^n$ of the system

$$\sum_{i=1}^{n} \beta_i b_i = -\sum_{i=n+1}^{N} \alpha_i b_i$$

has the property that $\beta_1, ..., \beta_n > 0$.

3. Again we may and shall assume that $a_1, ..., a_n$ are linearly independent, Then, since $(\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$ is the solution of the system

$$\sum_{i=1}^{n} \alpha_i a_i = -\alpha_{n+1} a_{n+1},$$

$\alpha_{n+1} \neq 0$ and, by Cramer's rule,

$$\alpha_i = \frac{-\alpha_{n+1}(-1)^{n+i}\det(a_1, ..., \widehat{a_i}, ..., a_{n+1})}{\det(a_1, ..., a_n)} \qquad \text{for} \quad i = 1, ..., n,$$

so that

$$\gamma = \frac{(-1)^{n+1}\alpha_{n+1}}{\det(a_1, ..., a_n)}.$$

A FACT FROM LINEAR ALGEBRA. *Consider these conditions on* G.

(a) *card* $Z(g) \leqslant n$ *for every* $g \in G \sim \{0\}$.

(b) *For any distinct points* $x_1, ..., x_{n+1} \in X$, $\det(v(x_1), ..., \widehat{v(x_i)}, ..., v(x_{n+1})) \neq 0$ *for some* $i \in \{1, ..., n+1\}$.

(c) *For any* $1 \leqslant m \leqslant n$ *distinct points* $x_1, ..., x_m \in X$,

$$\dim\{g \in G : g(x_1) = \cdots = g(x_m) = 0\} = \begin{cases} n-m & \text{if } v(x_1), ..., v(x_m) \text{ are linearly independent} \\ n-m+1 & \text{otherwise.} \end{cases}$$

(d) *For any* $1 \leqslant m \leqslant n$ *distinct points* $x_1, ..., x_m \in X$, *if* $\sum_{i=1}^m \alpha_i v(x_i) = 0$ *for some* $\alpha_1, ..., \alpha_m \in \mathbf{R} \sim \{0\}$, *then any non-empty proper subfamily of* $\{v(x_i)\}_{i=1,...,m}$ *is linearly independent.*

Then (a) *is equivalent to* (b), (b) *implies* (c), *and* (c) *is equivalent to* (d).

Proof. (a) $\Leftrightarrow$ (b). This follows immediately from the observation that for any distinct $x_1, ..., x_{n+1} \in X$ and any $g = \sum_{i=1}^n c_i g_i \in G$,

$$g(x_1) = \cdots = g(x_{n+1}) = 0 \quad \text{iff} \quad \sum_{i=1}^n c_i g_i(x_j) = 0 \quad \text{for} \quad j = 1, ..., n+1.$$

(b) $\Rightarrow$ (c). If $x_1, ..., x_m \in X$ are distinct, and if $g = \sum_{i=1}^n c_i g_i \in G$, then

$$g(x_1) = \cdots = g(x_m) = 0 \quad \text{iff} \quad \sum_{i=1}^n c_i g_i(x_j) = 0 \quad \text{for} \quad j = 1, ..., m$$

and

$$\operatorname{rank}(v(x_1), ..., v(x_m)) \geqslant m-1$$

(the rank condition is obvious if card $X = n$; if card $X \geqslant n+1$, choose distinct $x_{m+1}, ..., x_{n+1} \in X \sim \{x_1, ..., x_m\}$ and note that (b) says just that $\operatorname{rank}(v(x_1), ..., v(x_{n+1})) = n$).

(c) $\Rightarrow$ (d). First note that (d) holds for trivial reasons if $m = 1$. Now, let $2 \leqslant m \leqslant n$ and let $\sum_{i=1}^m \alpha_i v(x_i) = 0$ for some distinct $x_1, ..., x_m \in X$ and some $\alpha_1, ..., \alpha_m \in \mathbf{R} \sim \{0\}$. For $i \in \{1, ..., m\}$,

$$v(x_i) = -\sum_{\substack{j=1 \\ j \neq i}}^m \frac{\alpha_j}{\alpha_i} v(x_j),$$

whence

$$\operatorname{rank}(v(x_1), ..., \widehat{v(x_i)}, ..., v(x_m)) = \operatorname{rank}(v(x_1), ..., v(x_m)),$$

and the latter rank, by (c), is $m-1$.

(d) $\Rightarrow$ (c). First note that (c) holds for trivial reasons if $m=1$. Now, let $2 \leqslant m \leqslant n$ and let $x_1, ..., x_m \in X$ be such that $v(x_1), ..., v(x_m)$ are linearly dependent. Part (d) implies that there is exactly one non-empty subset I of $\{1, ..., m\}$ with the property that $\sum_{i \in I} \alpha_i v(x_i) = 0$ for some non-zero real numbers α_i, $i \in I$: Were there two distinct such sets, say,

$$\sum_{i \in I_1} \alpha_{i,1} v(x_i) = 0 = \sum_{i \in I_2} \alpha_{i,2} v(x_i),$$

then

$$\varepsilon \sum_{i \in I_1} \alpha_{i,1} v(x_i) + \frac{1}{\varepsilon} \sum_{i \in I_2} \alpha_{i,2} v(x_i) = 0$$

for every $\varepsilon > 0$; and for ε sufficiently small,

$$\varepsilon \sum_{i \in I_1} |\alpha_{i,1}| < \frac{1}{\varepsilon} \inf_{i \in I_2} |\alpha_{i,2}|,$$

so that $\varepsilon \alpha_{i,1} + (1/\varepsilon) \alpha_{i,2} \neq 0$ for every $i \in I_1 \cap I_2$. This shows that I is indeed unique, and then it is clear that for each $i \in I$, the vectors

$$\{v(x_j)\}_{j \in \{1, ..., m\} \sim \{i\}}$$

are linearly independent.

THE UNIFORMITY OF X. The set

$$\mathscr{U} = \{U : U \text{ is a neighborhood of } \Delta_2 \text{ in } X^2\}$$

is the unique compatible uniformity of X, in the sense that for every $x \in X$ the set

$$\{U[x] : U \in \mathscr{U}\}$$

is the neighborhood filter of x; here

$$U[x] = \{y \in X : (x, y) \in U\} \qquad \text{for every } U \in \mathscr{U} \text{ and every } x \in X.$$

This can be found, for example, in J. L. Kelley [7]. All other uniform notions employed in the present paper can also be found there, with the exception of the following.

For $U \in \mathscr{U}$, a subset Y of X is a *U-net in X* if

$$X = \bigcup_{y \in Y} U[y].$$

By compactness, for every $U \in \mathscr{U}$, there exists a finite U-net in X.

For a non-empty equicontinuous subset F of $C(X)$, the *joint modulus of continuity of F* is the function defined by

$$\Omega(F; U) = \sup\{\sup\{|f(x) - f(y)| : (x, y) \in U\} : f \in F\}, \qquad U \in \mathscr{U}.$$

By uniform continuity, if $\mathscr{U}$ is directed by

$$U \leqslant V \qquad \text{if} \quad U \supset V,$$

then, for every non-empty equicontinuous subset F of $C(X)$, the net $\{\Omega(F; U)\}_{U \in \mathscr{U}}$ converges to zero, in symbols,

$$\lim_{U \in \mathscr{U}} \Omega(F; U) = 0.$$

The following two discretization lemmas are adaptations of results of E. W. Cheney [2].

THE FIRST DISCRETIZATION LEMMA. *Let $f \in C(X) \sim G$, let $\{U_k\}_{k \in \mathbf{N}}$ be a sequence in $\mathscr{U}$ such that*

$$\Omega(f, g_1, ..., g_n; U_k) \leqslant 1/k \qquad \textit{for} \quad k = 1, 2, ...,$$

and for each $k \in \mathbf{N}$, let Y_k be a finite U_k-net in X and $h_k \in P_{Y_k}(f)$.

Then the sequence $\{d_{Y_k}(f)\}_{k \in \mathbf{N}}$ converges to $d(f)$, and the sequence $\{h_k\}_{k \in \mathbf{N}}$ is uniformly bounded on X and all of its cluster points belong to $P(f)$.

Proof. Since $G + \mathbf{R}f$ is an $(n+1)$-dimensional subspace of $C(X)$,

$$\gamma = \sup\left\{|c| + \sum_{i=1}^{n} |c_i| : \left\| cf + \sum_{i=1}^{n} c_i g_i \right\| \leqslant 1\right\} < \infty.$$

Fix $k \in \mathbf{N}$ such that $k > \gamma$, and fix $g = \sum_{i=1}^{n} c_i g_i \in G$. Choose $x \in X$ such that $|(f-g)(x)| = \|f - g\|$ and choose $y \in Y_k$ such that

$$\sup\{|f(x) - f(y)|, |g_1(x) - g_1(y)|, ..., |g_n(x) - g_n(y)|\} \leqslant 1/k.$$

Then

$$\begin{aligned}
\frac{1}{\gamma}&\left(1+\sum_{i=1}^{n}|c_i|\right)\\
&\leqslant \|f-g\| = |(f-g)(x)| \leqslant |(f-g)(x)-(f-g)(y)| + |(f-g)(y)|\\
&\leqslant |f(x)-f(y)| + \sum_{i=1}^{n}|c_i|\,|g_i(x)-g_i(y)| + |(f-g)(y)|\\
&\leqslant \frac{1}{k}\left(1+\sum_{i=1}^{n}|c_i|\right) + \|f-g\|_{Y_k}.
\end{aligned}$$

It follows that

$$\left(\frac{1}{\gamma}-\frac{1}{k}\right)\left(1+\sum_{i=1}^{n}|c_i|\right) \leqslant \|f-g\|_{Y_k},$$

and with this that

$$\|f-g\| \leqslant \left(1+\frac{\gamma}{k-\gamma}\right)\|f-g\|_{Y_k}.$$

This shows that if $k>\gamma$, then

$$\begin{aligned}
d(f) &\leqslant \|f-h_k\| \leqslant \left(1+\frac{\gamma}{k-\gamma}\right)\|f-h_k\|_{Y_k} = \left(1+\frac{\gamma}{k-\gamma}\right) d_{Y_k}(f)\\
&\leqslant (1+\frac{\gamma}{k-\gamma}\Big)\, d(f) \xrightarrow[k\to\infty]{} d(f),
\end{aligned}$$

whence $\{d_{Y_k}(f)\}_{k\in\mathbf{N}}$ and $\{\|f-h_k\}_{k\in\mathbf{N}}$ converge to $d(f)$. This does it.

THE SECOND DISCRETIZATION LEMMA. *Let $f\in C(X)\sim G$, let $Y_1\subset Y_2\subset\cdots$ be an increasing sequence of finite subsets of X such that* $\dim(G\mid Y_1)=n$, *and for each $k\in\mathbf{N}$, let $h_k\in P_{Y_k}(f)$ and $y_k\in Y_{k+1}$ be such that*

$$|(f-h_k)(y_k)| \geqslant d_{Y_k}(f) + \beta(\|f-h_k\| - d_{Y_k}(f))$$

for some constant $\beta>0$.

Then the sequence $\{d_{Y_k}(f)\}_{k\in\mathbf{N}}$ converges (monotonically!) to $d(f)$, and the sequence $\{h_k\}_{k\in\mathbf{N}}$ is uniformly bounded on X and all of its cluster points belong to $P(f)$.

Proof. Since $\dim(G\mid Y_1)=n$,

$$\gamma = \sup\{\|g\| : \|g\|_{Y_1}\leqslant 1\} < \infty.$$

If $g \in G$ is such that $\|g\| > 2\gamma \|f\|$, then for every $k \in \mathbf{N}$,

$$\begin{aligned}\|f-g\|_{Y_k} &\geqslant \|f-g\|_{Y_1} \geqslant \|g\|_{Y_1} - \|f\|_{Y_1} \geqslant \|g\|_{Y_1} - \|f\| \\ &\geqslant \frac{1}{\gamma}\|g\| - \|f\| > 2\|f\| - \|f\| = \|f\| \\ &\geqslant \|f\|_{Y_k} \geqslant d_{Y_k}(f),\end{aligned}$$

and therefore $g \notin P_{Y_k}(f)$. This shows that the sequence $\{h_k\}_{k \in \mathbf{N}}$ is uniformly bounded on X. Let h be a cluster point of $\{h_k\}_{k \in \mathbf{N}}$ in G, say, $h = \lim_{l \in \mathbf{N}} h_{k_l}$, and set $r = \lim_{k \in \mathbf{N}} d_{Y_k}(f)$. Then

$$\begin{aligned} r \leqslant d(f) &\leqslant \|f-h\| \leqslant \|f-h_{k_l}\| + \|h_{k_l}-h\| \\ &\leqslant \frac{1}{\beta}\left(|(f-h_{k_l})(y_{k_l})| - d_{Y_{k_l}}(f)\right) + d_{Y_{k_l}}(f) + \|h_{k_l}-h\| \\ &\leqslant \frac{1}{\beta}\left(|(f-h_{k_{l+1}})(y_{k_l})| + |(h_{k_{l+1}}-h_{k_l})(y_{k_l})| - d_{Y_{k_l}}(f)\right) \\ &\quad + d_{Y_{k_l}}(f) + \|h_{k_l}-h\| \\ &\leqslant \frac{1}{\beta}\left(d_{Y_{k_{l+1}}}(f) + \|h_{k_{l+1}}-h_{k_l}\| - d_{Y_{k_l}}(f)\right) + d_{Y_{k_l}}(f) \\ &\quad + \|h_{k_l}-h\| \xrightarrow[k \to \infty]{} r,\end{aligned}$$

whence $r = d(f)$ and $h \in P(f)$. We are done.

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